How Much Integral Action Can a Control System Tolerate?

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ABSTRACT

A closed formula is derived for the largest amount of integral action that an integral controllable system can tolerate without losing closed-loop stability. A block-structured guardian-map approach is used. A connection is obtained with the calculation of the maximal stability range of a singularly perturbed system.

1. INTRODUCTION

Many industrial systems can be stabilized using integral action, which has the desirable property of guaranteeing zero steady-state errors. In practice, many systems are *integral controllable* [1, 2], meaning that closed-loop stability is maintained as the amount of integral action is turned up from zero. In that case, the fundamental open question is: As the amount of integral action is increased from zero, when is closed-loop stability first lost? We answer that question by deriving a closed formula for the "radius of integral controllability," under mild conditions.

The approach taken is to apply recent work on guardian maps [3]. A key step is the judicious choice of a suitable guardian map using *block* Kronecker algebra [4]. By preserving the natural block structure of the closed-loop A-matrix, the analysis can be taken all the way to a closed eigenvalue formula.

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A by-product of the analysis is a connection with the problem [5] of determining the maximum stability range of a singularly perturbed system.

Closely related to the present work are results [6, 7] on the stability of a convex or linear combination of a stable matrix and some other matrix. However, those results are not directly applicable to the case in hand, because the closed-loop A-matrix is unstable in the absence of integral action.

2. RADIUS OF INTEGRAL CONTROLLABILITY

Consider an $m \times m$ system G with minimal transfer function matrix $G(s) = D + C(sI - A)^{-1}B$, where A is $n \times n$ and stable. Assume that G is *integral controllable* [1, 2]. That is, there exists $k^* > 0$ such that the integral controller kI_m/s stabilizes G in a negative feedback loop for all $k \in (0, k^*)$, where I_m is the $m \times m$ identity matrix. Our goal is find $k^*_{\max}$, the largest possible $k^* > 0$ such that kI_m/s stabilizes G for all $k \in (0, k^*)$. In the sequel, $k^*_{\max}$ is referred to as the *radius of integral controllability*.

In the statement of the main result that now follows, $\lambda^+_{\min}(\cdot)$ denotes the smallest positive real eigenvalue of a square matrix, or $+\infty$ if there are no positive real eigenvalues. Also, $\otimes$ and $\oplus$ denote the usual Kronecker product and Kronecker sum [8], respectively.

THEOREM 1. *Let $G(s) = D + C(sI - A)^{-1}B$ be the transfer function of an n-state stable $m \times m$ system that is integral controllable. Assume that D and $-D$ have no eigenvalues in common, and that $\tilde{A}$ and $-\tilde{A}$ have no eigenvalues in common, where $\tilde{A} := A - BD^{-1}C$. Under these conditions, the radius of integral controllability is*

$$k^*_{\max} = \lambda^+_{\min}(Y),$$

where Y is the $2mn \times 2mn$ matrix

$$Y := \begin{bmatrix} D^{-1} \otimes A & 0 \\ 0 & A \otimes D^{-1} \end{bmatrix} + \begin{bmatrix} D^{-1}C \otimes \tilde{A} & I_m \otimes B \\ \tilde{A} \otimes D^{-1}C & B \otimes I_m \end{bmatrix}$$

$$\times \begin{bmatrix} (\tilde{A} \oplus \tilde{A})^{-1} & 0 \\ 0 & -(D \oplus D)^{-1} \end{bmatrix} \begin{bmatrix} BD^{-1} \otimes I_n & I_n \otimes BD^{-1} \\ D^{-1} \otimes C & C \otimes D^{-1} \end{bmatrix}.$$

Proof. Connect kI_m/s to $G(s)$ in a negative feedback loop. The closed-loop A-matrix is

$$\bar{A} := \begin{bmatrix} A & -Bk \\ C & -Dk \end{bmatrix}.$$

Define

$$\nu(k) := \det(\bar{A} \mathbin{\bar{\oplus}} \bar{A})$$

where $\bar{\oplus}$ is the *block* Kronecker sum defined in [4]. According to [4], the set of eigenvalues of $\bar{A} \mathbin{\bar{\oplus}} \bar{A}$ is the set of all pairwise sums of eigenvalues of $\bar{A}$. So, if $\bar{A}$ has all its eigenvalues in the closed left half complex plane, then $\nu(k) = 0$ if and only if $\bar{A} \mathbin{\bar{\oplus}} \bar{A}$ is singular if and only if $\bar{A}$ has imaginary-axis eigenvalues. Hence $\nu(k)$ *guards* the open left half plane, in the sense defined in [3].

Integral controllability of G guarantees that for small enough $k > 0$ all the eigenvalues of $\bar{A}$ are in the open left half plane. Because $\nu(k)$ guards the open left half plane, as k then increases, the first value $k^*_{\max}$ for which closed-loop stability is lost is the smallest positive real root of $\nu(k) = 0$ (or $k^*_{\max} = +\infty$ if there are no positive real roots).

From the definition of $\bar{\oplus}$ we have that

$$\nu(k) = \det\begin{bmatrix} A \oplus A & -k(I_n \otimes B) & -k(B \otimes I_n) & 0 \\ I_n \otimes C & (A \otimes I_m) - k(I_n \otimes D) & 0 & -k(B \otimes I_m) \\ C \otimes I_n & 0 & (I_m \otimes A) - k(D \otimes I_n) & -k(I_m \otimes B) \\ 0 & C \otimes I_m & I_m \otimes C & -k(D \oplus D) \end{bmatrix}.$$

By assumption A is stable, so $A \oplus A$ is nonsingular, because [8] the set of eigenvalues of $A \oplus A$ is the set of all pairwise sums of eigenvalues of A. Similarly, $D \oplus D$ is nonsingular because, by assumption, D and $-D$ have no common eigenvalues. One can therefore use the Schur formula to evaluate the partitioned determinant, to obtain

$$\nu(k) = \det(A \oplus A)\det[-k(D \oplus D)]\det(L - kM),$$

where the $2mn \times 2mn$ matrices L and M are

$$L := \begin{bmatrix} I_m \otimes A & 0 \\ 0 & A \otimes I_m \end{bmatrix} - \begin{bmatrix} I_m \otimes B \\ B \otimes I_m \end{bmatrix} (D \oplus D)^{-1} \begin{bmatrix} I_m \otimes C & C \otimes I_m \end{bmatrix}$$

and

$$M := \begin{bmatrix} D \otimes I_n & 0 \\ 0 & I_n \otimes D \end{bmatrix} - \begin{bmatrix} C \otimes I_n \\ I_n \otimes C \end{bmatrix} (A \oplus A)^{-1} \begin{bmatrix} B \otimes I_n & I_n \otimes B \end{bmatrix}.$$

Some more determinantal manipulation leads to

$$\det M = \det D^{2n} \det(\tilde{A} \oplus \tilde{A}) \det(A \oplus A)^{-1},$$

where $\tilde{A} := A - BD^{-1}C$. By assumption $\tilde{A}$ and $-\tilde{A}$ have no common eigenvalues, so $\tilde{A} \oplus \tilde{A}$ is nonsingular, and hence so is M.

The roots of $\nu(k) = 0$ are therefore $k = 0$ together with the eigenvalues of LM^{-1}. Hence the smallest positive real root of $\nu(k) = 0$ is $\lambda^{+}_{\min}(LM^{-1})$.

It remains to show that $LM^{-1} = Y$. The first step is to use the matrix inversion lemma to write

$$M^{-1} = \begin{bmatrix} D^{-1} \otimes I_n & 0 \\ 0 & I_n \otimes D^{-1} \end{bmatrix} + \begin{bmatrix} D^{-1}C \otimes I_n \\ I_n \otimes D^{-1}C \end{bmatrix} (\tilde{A} \oplus \tilde{A})^{-1} \begin{bmatrix} BD^{-1} \otimes I_n & I_n \otimes BD^{-1} \end{bmatrix}.$$

The proof is completed by multiplying out LM^{-1} and collecting terms, simplifying where possible using standard properties of Kronecker algebra [8].

EXAMPLE 1. Consider the simplest case where G is a one-state SISO system that is integral controllable. That is, $G(s) = d + cb/(s - a)$, where $a < 0$, $d \neq 0$, and $d - cb/a > 0$. In that case

$$Y = \begin{bmatrix} a/d & 0 \\ 0 & a/d \end{bmatrix}.$$

Therefore

$$k^*_{\max} = \lambda^{+}_{\min}(Y) = \begin{cases} a/d & \text{if} \quad d < 0, \\ +\infty & \text{if} \quad d > 0. \end{cases}$$

In this very simple case, the result can be readily checked using the Routh-Hurwitz criterion. Simply note that the closed-loop characteristic equation is

$$\lambda^2 + (dk - a)\lambda + (bc - ad)k = 0.$$

By the Routh-Hurwitz criterion, for stability it is necessary and sufficient that $dk - a > 0$ and $(bc - ad)k > 0$, which leads to $k^*_{\max}$ as before.

REMARK 1. A close connection between integral controllability and stability of singularly perturbed systems is evident. The connection arises because $\bar{A}$, the closed-loop A-matrix, has the same inertia as

$$\begin{bmatrix} -D^T & -B^T \\ C^T/k & A^T/k \end{bmatrix}.$$

So $G(s)$ is stabilized by kI_m/s for all $k \in (0, k^*)$ if and only if the singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= -D^T x_1 - B^T x_2 \\ k\dot{x}_2 &= C^T x_1 + A^T x_2 \end{aligned}$$

is stable for all $k \in (0, k^*)$. In the latter case, the problem of finding $k^*_{\max}$ (the largest possible k^* for stability) has been solved in [5] using guardian-map theory. There the problem is reduced to one of finding the real roots of a polynomial in k. By using the block-structured approach of the present paper, the critical gain could be obtained directly from the solution of an eigenvalue problem, as the next example illustrates.

EXAMPLE 2. Consider the system with

$$A = \begin{bmatrix} -14.3 & 0 & -333 \\ 85.8 & -25 & -115 \\ 0 & 75 & -186 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1.6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0 & -275 \\ 0 & 0 & -56.9 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.2 & 0 \\ -0.5 & 0.5 \end{bmatrix}.$$

This system is taken from the example in Section 5 of [5], after applying the relationship of Remark 1 above to turn what was a singular perturbation problem into an equivalent integral controllability problem. The set of eigenvalues of Y is found to be two copies of

$$\{-1945, -654 \pm 256j, -12.3 \pm 104j, 67.26\}.$$

Hence $k^*_{\max} = 67.26$, which agrees with [5]. It is interesting to note that Y has only repeated eigenvalues. Based on numerical experience, we conjecture that this always occurs.

3. CONCLUSION

Under mild conditions, a closed formula was derived for the largest amount of integral action that an integral controllable system can tolerate before closed-loop stability is lost. This "radius of integral controllability" was shown to be the smallest positive real eigenvalue of a certain matrix, constructed from the state-space matrices of the plant. The solution technique was based on the construction and analysis of an appropriate block-structured guardian map. A connection with the problem of determining the maximal stability range of a singularly perturbed system was noted.

The technique can be readily extended to deal with other control laws. For example, see [9] for the case of proportional-plus-integral control.

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